

Opinion formation with upper and lower bounds

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Abstract. - We investigate the opinion formation with upper and lower bounds. We formulate the binary exchange of opinions between two individuals, and effects of the self-thinking and political party using the relativistic Boltzmann-Vlasov type equation with the randomly perturbed motion. The convergent form of the distribution function is determined by the balance between the cooling rate via the binary exchange of opinions between two individuals and the concentration of opinions by the political party, and heating rate via the self-thinking.

Introduction. – The opinion formation has been studied with great interests in the framework of the sociophysics [1]. In particular, the microscopic model proposed by Ben-Naim and his coworkers successfully demonstrated the opinion formation [2]. Meanwhile, kinetic models of the opinion formation have been discussed by Toscani and his coworkers [3] using the inelastic Boltzmann equation or one dimensional partial differential equation (PDE) with the strongly nonlinear form. The one parameter (m), which expresses the strength of the opinion on the single issue, is introduced. As a result, the positive value of m corresponds to the agreement on the single issue, whereas the negative value of m corresponds to the disagreement on the single issue. The binary exchange (collision) of opinions between two individuals with the opinion m_1 and m_2 cannot be expressed by the one dimensional elastic Boltzmann equation, because $m_1 \rightarrow m_2$ and $m_2 \rightarrow m_1$ never change the distribution function of m , namely, $f(m)$. Meanwhile, the compromise of opinions between two individuals is described by the perfectly inelastic collision. The successive inelastic collisions indicate that opinions of all the individuals converge to the averaged value of opinions (\bar{m}), as the time passes. Such a convergence of m via the binary exchange of opinions between two individuals is implausible in a real society. Then, we assume that the self-thinking occurs, when the binary exchange of opinions between two individuals occurs. To express effects of the self-thinking, the randomly perturbed motion [4] is incorporated into the binary collisional process. Such a perturbed motion at the binary collision corresponds to the Brownian motion under the thermal bath. As a result, effects of the self-thinking cause the diffusion of m , which prevents the convergence of m to the \bar{m} via binary inelastic collisions. In previous studies by Ben-Naim [2], the strength of the opinion was not bounded, such as $-\infty \leq m \leq \infty$, whereas the strength of the opinion was bounded by $|m| \leq 1$ in

previous studies by Toscani *et al.* [3]. In the real society, the individual with the infinitely strong opinion might be implausible, because the complete agreement and disagreement with the single issue must set the upper and lower bounds to m . Meanwhile, the PDE of m proposed by Toscani *et al.* [3] can not include the diffusion of m via the self-thinking, because the diffusion yields $1 < |m|$. Then, we propose the relativistic kinetic model, which always satisfies $|m| \leq 1$.

Relativistic kinetic model. – In this letter, the binary exchange of opinions between two individuals and self-thinking are expressed by the relativistic inelastic Boltzmann equation with the randomly perturbed motion. Additionally, we add the Vlasov term via the political party, which concentrates opinions of individuals to the opinion of the political party, namely, m_p . Finally, the relativistic kinetic model is formulated as

$$\begin{aligned} & \frac{\partial f(t, p)}{\partial t} \\ &= A \int_{-\infty}^{\infty} \left[\frac{1}{\mathcal{J}} f(t, p'') f(t, p'_*) - f(t, p) f(t, p_*) \right] g_{\phi} dp_* \\ &+ B \frac{\partial (p - P) f(t, p)}{\partial p}, \end{aligned} \quad (1)$$

where $f(t, p)$ is the distribution function, in which p and p_* are momentums of two colliding opinions, which are defined by $p = \gamma(m)m$ and $p_* = \gamma(m_*)m_*$ ($\gamma(m) = 1/\sqrt{1-m^2}$: Lorentz factor), and t is the time. The first term in the right hand side of Eq. (1) corresponds to relativistic inelastic collisions with the randomly perturbed motion, where g_{ϕ} is Møller's relative velocity [5], whereas the second term in the right hand side of Eq. (1) corresponds to the term by the political party, in which $P = m_p \gamma(m_p)$. The rate of the binary exchange of opinions between two individuals and rate of the concentration of m to m_p via the political party are expressed by A and B in Eq. (1), respectively. As a result of the direct inelastic collision with the randomly perturbed motion, momentums of two colliding opinions, namely, p and p_* , change to p' and p'_* , which are defined by $p' = p + \frac{1+\alpha}{2} (p_* - p + \Delta(p, p_*))$ and $p'_* = p_* - \frac{1+\alpha}{2} (p_* - p + \Delta(p, p_*))$, where α is the inelasticity coefficient ($0 \leq \alpha \leq 1$), and Δ is the randomly perturbed motion via the self-thinking. On the other hand, momentums of two colliding opinions, namely, p'' and p''_* , change to p and p_* , in which p'' and p''_* are defined by $p'' = p + \frac{1+\alpha}{2\alpha} (p_* - p + \Delta(p, p_*))$ and $p''_* = p_* - \frac{1+\alpha}{2\alpha} (p_* - p + \Delta(p, p_*))$. Consequently, the total momentum is conserved by the binary inelastic collision with the randomly perturbed motion, whereas the total energy ($E + E_* = \sqrt{1+p^2} + \sqrt{1+p_*^2}$) is not conserved by the binary collision with the randomly perturbed motion. In this letter, we restrict ourselves to $\alpha = 0$, which corresponds to the compromise of two colliding opinions [7], when $\Delta = 0$. Finally, \mathcal{J} in Eq. (1) is the Jacobian, which is defined by $J \equiv |\det(\partial(p'', p''_*)/\partial(p, p_*))|^{-1} = |1/\alpha + 1/2(1 + 1/\alpha)(\partial_{p_*} - \partial_p)\Delta(p, p_*)|^{-1}$.

The significant parameter in the opinion formation is the temperature (θ) in the closed opinion system, because $\theta \rightarrow \infty$ means that all $|m| \rightarrow 1$, where $|m| = 1$ corresponds to the complete agreement or disagreement on the single issue, namely, complete decision making. In our relativistic kinetic model, we never postulate the massless particle. Therefore, the individual with the complete decision making, namely, $|m| = 1$, is not considered. We, however, have a question, *What is the temperature in the closed opinion system?* The possible answer to this question is that the temperature in the closed opinion system is equivalent to the global interest in the single issue. Provided that all the individuals have high interests in the single issue, $|m|$ of all the individuals approximate to unity, namely, complete decision making. Meanwhile, m of all the individuals remain fuzzy state, namely, $|m| \ll 1$, when all the individuals have low interests in the single issue and the political party is absent. Finally, the global interest in the single issue decreases by the binary inelastic collision without the randomly perturbed motion (self-thinking), whereas the global interest increases by the self-thinking, namely, randomly perturbed motion Δ .

Numerical results. – In this letter, we investigate the opinion formation, which is described by the Eq. (1) by changing two parameters, Δ and B in Eq. (1), whereas A in Eq. (1) is fixed to the constant value, namely, $A = 1$. Additionally, physical quantities such as the density, averaged opinion (\bar{m}), and interest (θ) are calculated using Eckart's decomposition of $N^\alpha = \int_{-\infty}^{\infty} p^\alpha f dp / p^0$ and $T^{\alpha\beta} = \int_{-\infty}^{\infty} p^\alpha p^\beta f dp / p^0$ [8]. Finally, Eq. (1) is solved using the direct simulation Monte Carlo method [5] using 10^5 sample individuals.

At first, we investigate effects of Δ on the opinion formation by neglecting effects via the political party, namely, $B = 0$ in Eq. (1). The randomly perturbed motion is formulated as $\Delta = \Delta_a (2\mathcal{W} - 1)$, where Δ_a is the amplitude of the randomly perturbed motion, and $0 \leq \mathcal{W} \leq 1$ is the white noise. Provided that the binary collision is elastic, the global interest eternally increases via the self-thinking. As initial data, $f(0, m)$ is uniformly populated in the range of $0 \leq |m| < 1$. The time evolutions of the thermally relativistic measure $\chi = 1/k\theta$ (k : Boltzmann constant) and the decision making parameter defined by $\psi(t) \equiv \int_{-\infty}^{\infty} |p| f(t, p) dp$ are plotted in the left frame of Fig. 1, when $\Delta_a = 1$. χ (θ) temporally decreases (increases), whereas ψ temporally increases. Therefore, opinions of all the individuals approximate to ± 1 , namely, complete decision making via the self-thinking, as shown in the right frame of Fig. 1.

Next, we investigate the binary inelastic collision with the randomly perturbed motion (self-thinking), when $\Delta_a = 1.0, 5.0, 11.5$ and 25 under the absent of the political party, namely, $B = 0$ in Eq. (1). As initial data, $f(0, m)$ is uniformly populated in the range of $0.8 \leq |m| < 1$. Figure 2 shows snapshots of time evolutions of $f(t, m)$ versus m , which are obtained using $\Delta_a = 1.0, 5.0, 11.5$ and 25 . $f(1.26, m)$, which are obtained using $\Delta_a = 1.0, 5.0, 11.5$ and 25 , are similar to their convergent forms, namely, $f(\infty, m)$, which are determined by the balance between the cooling rate via the binary exchange of opinions between two individuals and heating rate via the self-thinking. As shown in the top-left frame of Fig. 2, $f(1.26, m)$ has its peak at $m = 0$, when $\Delta_a = 1.0$. Consequently, the cooling via the compromise ($\alpha = 0$) in the binary exchange of opinions between two individuals suppresses the heating via the self-thinking, when $\Delta_a = 1.0$. Meanwhile, opinions of individuals move toward the complete decision making ($m = \pm 1$), as Δ_a increases, as shown in top-right, bottom-left, and bottom right frames of Fig. 2. As shown in the top-right frame, $f(t, m)$ in the high opinion tail, namely, $|m| \sim 1$, temporally decreases, because the cooling via the compromise ($\alpha = 0$) in the binary exchange of opinions between two individuals suppresses the heating via the self-thinking. As shown in bottom-left and bottom-right frames, $f(t, m)$ in the high opinion tail, temporally increases. Figure 3 shows time evolutions of χ (left frame) and ψ (right frame), which are obtained using $\Delta_a = 1, 5, 11.5$ and 25 . As shown in the left frame of Fig. 3, global interests (θ), which are obtained using $\Delta_a = 1$ and 5 , markedly decreases and approximates to their convergent values, whereas θ obtained using $\Delta_a = 11.5$ slightly decreases and θ obtained using $\Delta_a = 25$ markedly increases and approximates to its convergent value. Similarly, decision making parameters (ψ), which are obtained using $\Delta_a = 1$ and 5 , markedly decrease and approximate to their convergent value, whereas ψ obtained using $\Delta_a = 11.5$ slightly decreases, and ψ obtained using $\Delta_a = 25$ markedly increases and approximates to its convergent value, as shown in the right frame of Fig. 3. Finally, convergent rates of χ and ψ increase, as Δ_a increases.

Next, we investigate the nonequilibrium state of $f(\infty, m)$ by comparing $f(\infty, m)$ with the equilibrium distribution function, namely, Maxwell-Jüttner function, $f_{MJ}(\infty, m)$, where we calculate $f(\infty, m)$ using the temporal average of $f(t, m)$, after $f(t, m)$ approximates to its convergent form. As discussed above, $f(t, m)$ approximates to its convergent form in accordance with Δ_a , when the cooling rate via the compromise and heating rate via the self-thinking are balanced. We, however, conjecture that $f(\infty, m)$ never approximates to $f_{MJ}(\infty, m)$, because the total collisional energy, namely, $E + E_*$, is not conserved in the inelastic binary collision with the randomly perturbed motion. Figure 4 shows $f(\infty, m)$ and $f_{MJ}(\infty, m)$ versus m , when $\Delta_a = 1$ (top-left frame), $\Delta_a = 5$ (top-right frame), $\Delta_a = 11.5$

(bottom-left frame) and $\Delta_a = 25$ (bottom-right frame). As shown in the top-left frame of Fig. 4, $f(\infty, m) \leq f_{MJ}(\infty, m)$ in the range of $0 \leq |m| \leq 0.3$, $f_{MJ}(\infty, m) \leq f(\infty, m)$ in the range of $0.3 \leq |m| \leq 0.58$ and $f(\infty, m) \leq f_{MJ}(\infty, m)$ in the range of $0.58 \leq |m| < 1$, when $\Delta_a = 1$. As shown in the top-right frame of Fig. 4, $f(\infty, m) \leq f_{MJ}(\infty, m)$ in the range of $0 \leq |m| \leq 0.72$, $f_{MJ}(\infty, m) \leq f(\infty, m)$ in the range of $0.72 \leq |m| \leq 0.95$ and $f(\infty, m) \leq f_{MJ}(\infty, m)$ in the range of $0.95 \leq |m| < 1$, when $\Delta_a = 5$. As shown in the bottom-left frame of Fig. 4, $f(\infty, m) \leq f_{MJ}(\infty, m)$ in the range of $0 \leq |m| \leq 0.9$, $f_{MJ}(\infty, m) \leq f(\infty, m)$ in the range of $0.9 \leq |m| \leq 0.98$ and $f(\infty, m) \leq f_{MJ}(\infty, m)$ in the range of $0.98 \leq |m| < 1$, when $\Delta_a = 11.5$. As shown in the bottom-right frame of Fig. 4, $f(\infty, m) \leq f_{MJ}(\infty, m)$ in the range of $0 \leq |m| \leq 0.97$ and $f_{MJ}(\infty, m) \leq f(\infty, m)$ in the range of $0.97 \leq |m| < 1$, when $\Delta_a = 25$. Finally, we can conclude that $f(\infty, m)$ is slightly different from the Maxwell-Jüttner function, as a result of binary inelastic collisions with the randomly perturbed motion.

Next, we investigate effects of the political party on the opinion formation. As initial data, $f(0, m)$ is uniformly populated in the range of $0.8 \leq |m| < 1$. We consider six cases, namely, $m_p = 0, 0.5$ and 0.8 , when $(\Delta_a, B) = (1, 0.1)$, and $m_p = 0, 0.5$ and 0.8 , when $(\Delta_a, B) = (11.5, 1)$. Figure 5 shows the convergent form of the distribution function, namely, $f(\infty, m)$, for $m_p = 0, 0.5$ and 0.8 , and $f_{MJ}(0, m)$ for $m_p = 0$ and $m_p = 0.5$, when $(\Delta_a, B) = (1, 0.1)$, in its left frame, and $f(\infty, m)$, for $m_p = 0, 0.5$ and 0.8 , and $f_{MJ}(0, m)$ for $m_p = 0$ and 0.5 , when $(\Delta_a, B) = (11.5, 1)$, in its right frame. $f_{MJ}(0, m)$ for $m_p = 0.8$ when $(\Delta_a, B) = (1, 0.1)$ and $(\Delta_a, B) = (11.5, 1)$, are not shown in Fig. 5, because the modified Bessel function of the second kind, which defines $f_{MJ}(0, m)$, cannot be calculated using χ , which is larger than 500, owing to the unestablished algorithm to solve the modified Bessel function with $500 < \chi$. As shown in left and right frames of Fig. 5, m moves toward m_p owing to the political party, which is expressed by Vlasov term in Eq. (1). The value of $f(\infty, m_p)$ increases, as m_p increases, as shown in the left and right frames of Fig. 5. Consequently, we obtain $f(\infty, m \leq -0.5) \ll 1$, when $(\Delta_a, B) = (1.0, 0.1)$ and $m_p = 0.5$, and $f(\infty, m \leq 0.5) \ll 1$, when $(\Delta_a, B) = (1.0, 0.1)$ and $m_p = 0.8$. On the other hand, $f(\infty, m)$ for $m_p = 0$ and 0.5 have finite values at $|m| \sim 1$ and $f(\infty, m)$ for $m_p = 0.8$ has the markedly sharp peak at $m = m_p$, when $(\Delta_a, B) = (11.5, 1)$. The sharp $f(\infty, m_p)$ and $f(t, 0)|_{m_p=0} < f(t, 0.5)|_{m_p=0.5} < f(t, 0.8)|_{m_p=0.8}$ in left and right frames of Fig. 5 are described by the fact that the solution of $dp/dt = -B(p - P)$, which is the characteristic equation of the Vlasov term in Eq. (1), indicates that m is damped to m_p more rapidly, as $|m - m_p|$ becomes smaller, and such a damping rate increases, as m_p increases. From above results, the strong opinion of the political party concentrates opinions of individuals to the opinion of the political party, whereas the neutral opinion of the political party allows opinions, which are different from that of the political party, when the heating via the self-thinking is large, adequately.

Figure 6 shows time evolutions of χ (top-left frame) and ψ (top-right frame) for $m_p = 0, 0.5$, and 0.8 , when $(\Delta_a, B) = (1.0, 0.1)$, together with time evolutions of χ and ψ , when $(\Delta_a, B) = (1.0, 0)$, and time evolutions of χ (bottom-left frame) and ψ (bottom-right frame) for $m_p = 0, 0.5$ and 0.8 , when $(\Delta_a, B) = (11.5, 1)$, together with time evolutions of χ and ψ , when $(\Delta_a, B) = (11.5, 0)$. As shown in the top-left frame of Fig. 6, the convergent value of the global interest ($\theta(\infty)$) decreases, as m_p increases. Meanwhile, $\theta(\infty)$, which are obtained using $m_p = 0$ and 0.5 , when $(\Delta_a, B) = (11.5, 1)$, are markedly lower than that obtained using $(\Delta_a, B) = (11.5, 0)$. Additionally, $\theta(\infty)$, which is obtained using $m_p = 0$, when $(\Delta_a, B) = (11.5, 1)$, is similar to that obtained using $m_p = 0.5$, when $(\Delta_a, B) = (11.5, 1)$, whereas $\theta(\infty)$, which is obtained using $m_p = 0.8$, when $(\Delta_a, B) = (11.5, 1)$, is markedly lower than those obtained using $m_p = 0$ and 0.5 , when $(\Delta_a, B) = (11.5, 1)$. The relation, $\theta(\infty)|_{m_p=0.8} \ll \theta(\infty)|_{m_p=0} \simeq \theta(\infty)|_{m_p=0.5}$, when $(\Delta_a, B) = (11.5, 1)$, implies that the drastic transition of $\theta(\infty)$ occurs at the critical value of m_p , when $\theta(\infty)$ is high enough such as $\chi(\infty) \sim 1$, whereas $\theta(\infty)$ decreases gradually, as m_p increases, when $\theta(\infty)$ is low such as $10 < \chi(\infty)$, as shown in the case of $(\Delta_a, B) = (1, 0.1)$. The top-right frame of Fig. 6

shows that the convergent value of the decision making parameter ($\psi(\infty)$), which is obtained using $m_p = 0$, when $(\Delta_a, B) = (1, 0.1)$, is smaller than that obtained when $(\Delta_a, B) = (1, 0)$, whereas $\psi(\infty)$ increases, as m_p increases, when $(\Delta_a, B) = (1, 0.1)$. As a result, we obtain the relation, $\psi(\infty)|_{m_p=0} < \psi(\infty)|_{B=0} < \psi(\infty)|_{m_p=0.5} < \psi(\infty)|_{m_p=0.8}$, when $\Delta_a = 1$ and $B = 0.1$ for $m_p = 0, 0.5$ and 0.8 . The stronger opinion of the political party yields the larger $\psi(\infty)$, when $(\Delta_a, B) = (1, 0)$. Such a tendency is obtained, when $(\Delta_a, B) = (11.5, 1)$, whereas we obtain the relation, $\psi(\infty)|_{m_p=0} < \psi(\infty)|_{m_p=0.5} < \psi(\infty)|_{m_p=0.8} < \psi(\infty)|_{B=0}$, when $\Delta_a = 11.5$ and $B = 1$ for $m_p = 0, 0.5$ and 0.8 .

Conclusions. — In this letter, we formulated the relativistic kinetic model in Eq. (1) to demonstrate the opinion formation with upper and lower bounds. The convergent form of the distribution function was determined by the balance between the cooling rate via the binary exchange of opinions between two individuals and the concentration of opinions by the political party, and heating rate via the self-thinking. In particular, the markedly strong opinion of the political party excluded individuals with opinions, which are different from the opinion of the political party. The decision making parameter becomes larger by the stronger opinion of the political party, even when the global interest becomes lower by the stronger opinion of the political party.

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Figure captions

Figure 1: Time evolutions of χ (y_1 axis) and ψ (y_2 axis) by the self-thinking (left frame). $f(t, m)$ versus m at $t = 0, 0.63$ and 1.26 by the self-thinking (right frame).

Figure 2: Snapshots of time evolution of $f(t, m)$ versus m , when $\Delta_a = 1$ (top-left frame), $\Delta_a = 5$ (top-right frame), $\Delta_a = 11.5$ (bottom-left frame) and $\Delta_a = 25$ (bottom-right frame).

Figure 3: Time evolutions of χ (left frame) and ψ (right frame), when $\Delta_a = 1, 5, 11.5$ and 25 .

Figure 4: $f(\infty, m)$ and $f_{MJ}(\infty, m)$ versus m , when $\Delta_a = 1$ (top-left frame), $\Delta_a = 5$ (top-right frame), $\Delta_a = 11.5$ (bottom-left frame) and $\Delta_a = 25$ (bottom-right frame).

Figure 5: $f(\infty, m)$ versus m , for $m_p = 0, 0.5$ and 0.8 , and $f_{MJ}(\infty, m)$ versus m for $m_p = 0$ and $m_p = 0.5$, when $(\Delta_a, B) = (1, 0.1)$, in left frame, and $f(\infty, m)$ versus m , for $m_p = 0, 0.5$ and 0.8 , and $f_{MJ}(\infty, m)$ versus m for $m_p = 0$ and 0.5 , when $(\Delta_a, B) = (11.5, 1)$, in right frame.

Figure 6: Time evolutions of χ (top-left frame) and ψ (top-right frame) for $m_p = 0, 0.5$ and 0.8 , when $(\Delta_a, B) = (1, 0.1)$ together with time evolutions of χ and ψ , when $(\Delta_a, B) = (1, 0)$. Time evolutions of χ (bottom-left frame) and ψ (bottom-right frame) for $m_p = 0, 0.5$ and 0.8 , when $(\Delta_a, B) = (11.5, 1)$ together with time evolutions of χ and ψ , when $(\Delta_a, B) = (11.5, 0)$.